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PROPAGATION OF ULTRASONIC WAVES IN POLYCRYSTALS OF CUBIC SYMMETRY WITH ALLOWANCE FOR MULTIPLE SCATTERING*

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Scattering coefficients and propagation velocities of longitudinal and transverse ultrasonic waves in polycrystals of cubic symmetry are calculated over the whole range of frequencies. The calculations are carried out in the Bourret approximation using methods of renormalization of the wave equation and re-expansion which takes into account multiple scattering with the exponential dependence on the correlation tensor coordinate. Asymptotics of low and high frequencies are determined, and numerical computations carried out for copper. The behavior of scattering coefficients determined here conforms to the known formulas for regions of the Rayleigh, phase, and diffusive scattering.

The propagation of ultrasonic waves in polycrystals is accompanied by their scattering over crystallites (inhomogeneity grains). A survey of publications on this subject appeared in /1/. This effect was first determined in /2/ in the Bourret approximation, using the method of wave equation renormalization for polycrystals of cubic symmetry. But only the asymptotics of long and short waves were considered under conditions of single scattering. The same method was used in /3/ for calculations over the whole range of wave lengths.

The multiple scattering coefficients and propagation velocities of ultrasonic waves over the whole range of frequencies and dimensions of inhomogeneity grains are determined below by methods of the statistical theory of elasticity.

1. Let us consider two bodies of the same dimensions and shape, one inhomogeneous whose effective dynamic modulus of elasticity is to be determined, the other a homogeneous reference body. The vector $\mathbf{u}$ of harmonic wave displacement in a medium defined by the tensor of elastic moduli $C_{nwrs}(\mathbf{r})$ satisfies the equation

$$L_{nr}u_r = 0, \quad L_{nr} \equiv \partial_w C_{nwrs}\partial_s + \rho\omega^2\delta_{nr}, \quad \partial_n \equiv \partial / \partial r_n \tag{1.1}$$

where ρ is the density of a medium which is assumed homogeneous, and ω is the angular frequency. We denote the quantities pertaining to the reference body by subscript c, those extraneous to that body by primes and, where possible, omit tensor indices.

Subtracting from Eq. (1.1) its value for the reference medium, we obtain

$$\mathbf{L}_c\mathbf{u}' = -\mathbf{L}\mathbf{u}, \quad \mathbf{L}' = \mathbf{L} - \mathbf{L}_c \tag{1.2}$$

Using the Green's tensor $\mathbf{G}$ of the regular operator $\mathbf{L}_c$, we obtain the solution of Eq. (1.1)

$$\mathbf{u}' = \mathbf{G} * \mathbf{L}'\mathbf{u} \tag{1.3}$$

where the asterisk denotes an integral convolution. We represent the second derivative of Green's tensor in the form of the sum of its singular $\mathbf{G}^{(s)}$ and formal $\mathbf{G}^{(f)}$ parts. For a nontextured polycrystal we select a spherical surface of the integration element /4/, and introduce a tensor g and the integral operator $\mathbf{p}$ defined by the equalities

$$g\mathbf{f} \equiv \mathbf{G}^{(s)} * \mathbf{f}, \quad \mathbf{p}\mathbf{f} \equiv \mathbf{G}^{(f)} * \mathbf{f} \tag{1.4}$$

where $\mathbf{f}$ is an arbitrary function. The first of equalities (1.4) is trivial, since the dependence of $\mathbf{G}^{(s)}$ on coordinates is $\delta(\mathbf{r})$.

We pass in (1.3) from the displacement vector to the strain tensor $\boldsymbol{\varepsilon}$, and assume that the reference body is large in comparison with the space correlation scale. Then

$$\boldsymbol{\varepsilon}' = (g + \mathbf{p})\,\mathbf{C}'\boldsymbol{\varepsilon} \tag{1.5}$$

We separate the local associated with $\mathbf{G}^{(s)}$ from the integral term which defines the nonlocal part of the interaction between inhomogeneities. We furthermore take into consideration that the contributions of local interactions between inhomogeneities can be exactly summed

$$\mathbf{e} = \boldsymbol{\varepsilon}_c + \mathbf{p}\mathbf{l}\mathbf{e}, \quad \mathbf{e} \equiv (\mathbf{I} - g\mathbf{C}')\,\boldsymbol{\varepsilon}, \quad \mathbf{l} = \mathbf{C}'(\mathbf{I} - g\mathbf{C}')^{-1} \tag{1.6}$$

$$\mathbf{e} = (\mathbf{I} - \mathbf{p}\mathbf{l})^{-1}\boldsymbol{\varepsilon}_c, \quad I_{nwrs} = {}^1/_2\,(\delta_{nr}\delta_{ws} + \delta_{ns}\delta_{wr}) \tag{1.7}$$

$$\mathbf{l}\mathbf{e} = \mathbf{C}'\boldsymbol{\varepsilon} \tag{1.8}$$

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Assuming that the tensor field of elasticity moduli is ergodic /5,6/, the averaging of equality (1.8) yields

$$\mathbf{l}_* \langle \mathbf{e} \rangle = \langle \mathbf{le} \rangle, \quad \mathbf{l}_* (\mathbf{I} - \mathbf{g}(\mathbf{C}_* - \mathbf{C}_c))^{-1} \langle \boldsymbol{\varepsilon} \rangle = (\mathbf{C}_* - \mathbf{C}_c) \langle \boldsymbol{\varepsilon} \rangle \tag{1.9}$$

From equalities (1.7) and (1.9) we have /7/

$$\mathbf{l}_* = \langle \mathbf{lR} \rangle, \quad \mathbf{R} \equiv (\mathbf{I} - \mathbf{pl})^{-1} \langle (\mathbf{I} - \mathbf{pl})^{-1} \rangle^{-1} = \sum_{n=0}^{\infty} (\mathbf{Hl})^n, \qquad \mathbf{H} \equiv (\mathbf{I} - \mathbf{M})\,\mathbf{p}, \quad \mathbf{Mf} \equiv \langle \mathbf{f} \rangle \tag{1.10}$$

Solution (1.10) represents an operator series whose determination requires the knowledge of multi-point moment functions of elastic constants. We set in (1.10) $\langle \mathbf{l} \rangle = 0$ and take into account only second order moments. This yields

$$\mathbf{l}_* = \langle \mathbf{lpl} \rangle \tag{1.11}$$

Condition $\langle \mathbf{l} \rangle = 0$ improves the convergence of series (1.10) and simplifies the calculation of $\mathbf{l}_*$. It simultaneously determines the constants of the reference body in the self-consistency approximation. In fact, if we assume that

$$\langle \mathbf{l} \rangle = \langle (\mathbf{C} - \mathbf{C}_c)(\mathbf{I} - \mathbf{g}(\mathbf{C} - \mathbf{C}_c)^{-1}) \rangle = 0 \tag{1.12}$$

then, introducing tensor $\mathbf{b}$ by the equality $\mathbf{b} = -\mathbf{C}_c - \mathbf{g}^{-1}$, /4/we obtain the formula

$$\mathbf{C}_c = \langle (\mathbf{C} + \mathbf{b})^{-1} \rangle^{-1} - \mathbf{b} \tag{1.13}$$

which, as shown in /8/, determines constants $\mathbf{C}_c$ in the self-consistency approximation.

Equations similar to (1.9) and (1.11) are also obtained by the summation of the Feynman diagram /9,10/.

The mean strain of harmonic waves depends on coordinates, which makes the direct determination of $\mathbf{C}_*$ by Eq. (1.9) impossible. However the application of the Fourier transform to that equation and the elimination of the transform $\langle \boldsymbol{\varepsilon} \rangle$ makes it possible to express the Fourier image $\bar{\mathbf{C}}_*$ of operator $\mathbf{C}_*$ in terms of the Fourier image $\bar{\mathbf{l}}_*$ of operator $\mathbf{l}_*$

$$\bar{\mathbf{C}}_* - \mathbf{C}_c = (\mathbf{I} + \bar{\mathbf{l}}_* \mathbf{g})^{-1} \bar{\mathbf{l}}_* \tag{1.14}$$

2. Let us determine $\bar{\mathbf{l}}_*$ on the assumption that the tensor A_{imrs}^{nwuv} and the coordinate $\varphi(\mathbf{r})$ functions have been separated. We have

$$\bar{\mathbf{l}}_{*imuv} = A_{imrs}^{nwuv} J_{nwrs}, \quad \langle l_{nwuv}(\mathbf{r})\, l_{imrs}(\mathbf{r}') \rangle = A_{imrs}^{nwuv} \varphi(\mathbf{r} - \mathbf{r}') \tag{2.1}$$

where the integral

$$J_{nwrs} = \int G_{n)(r,\,s)(w,}^{(f)}(\mathbf{r}, \omega)\, \varphi(\mathbf{r}) \cos(\mathbf{qr})\, d\mathbf{r}, \quad (\cdot)_{,n} \equiv \frac{\partial(\cdot)}{\partial r_n} \tag{2.2}$$

where $\mathbf{q}$ is the wave vector for polycrystals $\varphi(\mathbf{r}) = \exp(-r/a)$ /11/ and a is the correlation scale whose order of magnitude is the same as that of the inhomogeneity grain. Symmetrization is carried out with respect to indices appearing in parentheses.

The substitution of variables $\mathbf{q} = q\mathbf{k}$, $\boldsymbol{\zeta} = q\mathbf{r}$, and $\boldsymbol{\zeta}/\zeta = \mathbf{n}$ yields

$$J_{nwrs} = \int G_{n)(r,\,s)(w}^{(f)} \zeta^2 \exp(-x\zeta) \cos(\mathbf{k}\boldsymbol{\zeta})\, d\zeta\, d\Omega \tag{2.3}$$

$$x \equiv 1/(qa), \quad q = \omega/c \tag{2.4}$$

$$d\Omega \equiv d\boldsymbol{\zeta}/\zeta^2 d\zeta, \quad k_{nwr\ldots} \equiv k_n k_w k_r \ldots$$

$$4\pi\rho c^2 \zeta^3 G_{n)(r,\,s)(w}^{(f)}(\zeta, c, \eta) = h_1 k_{nwrs} + h_2 \varphi_{nwrs} + h_3 \delta_{nwrs} + h_4 \theta_{nwrs} + h_5 \delta_{n(r}\delta_{s)w} \tag{2.5}$$

$$\theta_{nwrs} \equiv k_{(n}\delta_{w)(r}k_{s)}, \quad \varphi_{nwrs} \equiv 4\theta_{nwrs} + \delta_{nw}k_{rs} + k_{nw}\delta_{rs}$$

$$\delta_{nwrs} \equiv \delta_{nw}\delta_{rs} + \delta_{nr}\delta_{ws} + \delta_{ns}\delta_{wr}$$

where c is the wave phase velocity in the reference body. Formulas (2.3)–(2.5) are valid for longitudinal and transverse waves. In the first case all quantities must be expressed in terms of c_l, q_l, x_l, and ζ_l, and in the second in terms of c_t, q_t, x_t, and ζ_t. For functions $h_n^{(l)}$ and $h_n^{(t)}$ we have

$$\zeta^2 h_1^{(l)} = d_1(\zeta\eta) - d_1(\zeta), \quad \zeta^2 h_2^{(l)} = d_2(\zeta) - d_2(\zeta\eta) \tag{2.6}$$

$$\zeta^2 h_3^{(l)} = d_3(\zeta\eta) - d_3(\zeta), \quad h_4^{(l)} = \eta^2 d_3(\zeta\eta), \quad h_5^{(l)} = \eta^2 d_4(\zeta\eta)$$

$$\zeta^2 h_1^{(t)} = d_1(\zeta) - d_1(\zeta\eta), \quad \zeta^2 h_2^{(t)} = d_2(\zeta\eta) - d_2(\zeta) \tag{2.7}$$

$$\zeta^2 h_3^{(t)} = d_3(\zeta) - d_3(\zeta\eta), \quad h_4^{(t)} = d_3(\zeta), \quad h_5^{(t)} = d_4(\zeta)$$

$$d_1(\varkappa) = (105 + 105\, i\varkappa - 45\varkappa^2 - 10\, i\varkappa^3 + \varkappa^4)\, e^{-i\varkappa}$$

$$d_2(\varkappa) = (15 + 15\, i\varkappa - 6\varkappa^2 - i\varkappa^3)\, e^{-i\varkappa}$$

$$d_3(\varkappa) = (3 + 3\, i\varkappa - \varkappa^2)\, e^{-i\varkappa}, \quad d_4(\varkappa) = -(1 + i\varkappa)\, e^{-i\varkappa}$$

For simplicity the subscripts l and t at ζ and η have been omitted in formulas (2.6) and (2.7), respectively. The ratios $\eta_l = 1/\eta_t = c_l/c_t$ are denoted by η, and the wave propagation velocities in the reference medium are determined by the bulk and shear moduli K_c and μ_c

$$c_l = [(K_c + 4\,\mu_c / 3) / \rho]^{1/2},\ c_t = (\mu_c / \rho)^{1/2} \tag{2.8}$$

The comparison of formulas (2.6) and (2.7) shows the following relationship between functions $h_n^{(l)}$ and $h_n^{(t)}$:

$$\begin{aligned} h_n^{(t)}(\zeta_t, c_t, \eta_t) &= -\,h_n^{(l)}(\zeta_t, c_t, \eta_t),\ n = 1, 2, 3 \\ h_n^{(t)}(\zeta_t, c_t) &= h_n^{(l)}(\zeta_t, c_t, 1),\ n = 4, 5 \end{aligned} \tag{2.9}$$

which enables us to calculate the integral (2.3) for the case of transverse waves using its expression for longitudinal waves.

Using formulas

$$\begin{aligned} T_{nwrs} &\equiv \int n_{nwrs} \cos(\zeta\mathbf{k})\, d\Omega = 4\pi\,(k_{nwrs} j_4 - \varphi_{nwrs} j_3 \zeta^{-1} + \delta_{nwrs} j_2 \zeta^{-2}) \\ T_{nw} &\equiv T_{nwrr} = 4\pi\,(\delta_{nw} j_1 \zeta^{-1} - k_{nw} j_2),\quad T \equiv T_{nn} = 4\pi j_0 \end{aligned} \tag{2.10}$$

where $j_n(\zeta)$ are spherical Bessel functions, with respect to angles, and obtain

$$J_{nwrs} = \int_0^{\infty} \zeta^2 e^{-\zeta x}\, [h_1 T_{nwrs} + 2h_2\,(T_{n(w}\delta_{r)s} + T_{w(r}\delta_{s)n} + T_{s(r}\delta_{n)w}) + h_3\delta_{nwrs} T + h_4 T_{(n}\delta_{w)(r} T_{s)} + h_5 \delta_{n(r}\delta_{s)w}]\, d\zeta \tag{2.11}$$

Integration of the expression (2.11) with respect to the variable ζ is carried out separately for longitudinal and transverse waves.

3. Let us first consider longitudinal waves. To simplify presentation of quantities such as $c, q, x,$ and η we omit the subscript l. We use the expressions for T in (2.10) and the explicit form of functions $h_n^{(l)}$ for longitudinal waves as defined in (2.6). Then formula (2.11) enables us to represent the sought integral J_{nwrs} in the form

$$J_{nwrs} = P_1 k_{nwrs} + P_2 \varphi_{nwrs} + P_3 \delta_{nwrs} + P_4 k_{(n}\delta_{w)(r} k_{s)} + P_5 \delta_{n(r}\delta_{s)w} \tag{3.1}$$

$$\rho\, c^2 P_n \equiv R_n(\eta, \beta) - Q_n(\alpha) \tag{3.2}$$

$$\begin{aligned} &R_1 = 35\, R_3 + 10\, R_5 + \eta^4 I_0^1,\ R_2 = -\,5\, R_3 - R_5,\ R_3 = \\ &\quad 3\, I_4^1 + 3\, i\, \eta\, I_4^2 - \eta^2\,(9\, I_3^1 - I_2^1) - 2\, i\, \eta^3 I_3^2 + \eta^4 I_2^1 \\ &R_4 = -\,3\, R_5 - \eta^4 I_0^1,\ R_5 = \eta^2 I_2^1 + i\, \eta^3 I_2^2 + \eta^4 I_1^1 \end{aligned} \tag{3.3}$$

$$I_n^m = \int_0^{\infty} e^{-\beta\zeta} j_n(\zeta)\, \zeta^{m-n}\, d\zeta \quad \begin{pmatrix} \beta = x + i\eta \\ \alpha = x + i \end{pmatrix} \tag{3.4}$$

Integrals I_n^m are calculated similarly to the Hankel integral /12/, and are defined by the hypergeometric function

$$I_n^m = \frac{m!\,\beta^{-m-1}}{(2n+1)!!}\, F\left(\frac{m+1}{2}, \frac{m+2}{2}; n + \frac{3}{2}; -\beta^{-2}\right) \tag{3.5}$$

In formula (3.4) m and n are nonnegative integers, and, in conformity with (2.4), $x = \operatorname{Re}\alpha = \operatorname{Re}\beta$ is positive. Hence the integral I_n^m exists throughout the whole region of admissible values of x, and $F(a, b; c; \zeta)$ is an elementary function of ζ. Furthermore, any three functions of the type $F(a + p_1,\ b + p_2;\ c + p_3;\ \zeta)$, where $p_1, p_2,$ and p_3 are integers $(c \neq 0, -1, -2, \ldots,)$ linked by a linear homogeneous function whose coefficients are polynomials/12/.

A general formula similar to the Christoffel formula can be derived for the integrals (3.5) /12/, but it is more convenient to use recurrent relations

$$\begin{aligned} &I_n^2 = \beta^{-1}\,[I_n^1 - (2\,n + 1)\, I_{n-1}^1] \\ &2\,n\,(2\,n + 1)\, I_{n+1}^1 = [\beta^2\,(2\,n + 1) + (4\,n - 1)]\, I_n^1 - (1 + \beta^2)\, I_{n-1}^1 \\ &I_1^1 = 1 - \beta \operatorname{arctg} \beta^{-1},\ I_0^1 = 1 / (1 + \beta^2) \end{aligned} \tag{3.6}$$

which enable us to reduce the sought integral to integrals I_1^1 and I_0^1. Formulas (3.6) are obtained from Gauss relationships for contiguous hypergeometric functions /12/ and the known representations of elementary functions

$$\beta^2 / (1 + \beta^2) = F(1, 3/2; 3/2; -\beta^{-2}),\quad \beta \operatorname{arctg} \beta^{-1} = F(1, 1/2; 3/2; -\beta^{-2})$$

Calculations by formulas (3.3)–(3.6) yield

$$\begin{aligned} &R_1 = 35\, R_3 + 10\, R_5 + \eta^4 / (1 + \beta^2),\ R_2 = -5\, R_3 - R_5 \\ &48\, R_3 = 48/35 + \beta^2\,(33/5 + 8\,\beta^2 + 3\,\beta^4) - 3\,\beta\,(1 + \end{aligned} \tag{3.7}$$

$$3\,\beta^2 + 3\,\beta^4 + \beta^6)\,J - i\,\eta\beta\,(81/5 + 38\,\beta^2 + 21\,\beta^4) + 3\,i\,\eta\,(1 + 9\,\beta^2 + 15\,\beta^4 + 7\,\beta^6)\,J - 6\,\eta^2\,(32/15 + 11\,\beta^2 + 9\,\beta^4) + 6\,\eta^2\beta\,(5 + 14\,\beta^2 + 9\,\beta^4)\,J + 4\,i\,\eta^3\beta\,(13 + 5\,\beta^2) - 12\,i\,\eta^3\,(1 + 6\,\beta^2 + 5\,\beta^4)\,J + 8\,\eta^4\,(2 + 3\,\beta^2) - 24\,\eta^4\beta \times (1 + \beta^2)\,J,\ R_4 = -3\,R_5 - \eta^4 / (1 + \beta^2),\ 2\,R_5 = \eta^2\,(2/3 + \beta^2) - \eta^2\beta\,(1 + \beta^2)\,J - 3\,i\,\eta^3\beta + i\,\eta^3\,(1 + 3\,\beta^2)\,J + 2\,\eta^4 \times (J\,\beta - 1),\ J \equiv \operatorname{arctg}\beta^{-1}$$

Corresponding expressions for $Q_n(\alpha)$ $(n = 1, 2, 3)$ are obtained from $R_n(\eta, \beta)$ by carrying out in (3.7) the substitutions $\beta \to \alpha$ and $\eta \to 1$, when $Q_4 = Q_5 = 0$.

The complex variables α and β are determined by the expressions in parentheses in (3.4). We pass from these variables to variables η_l and x_l, then substitute the result into (3.2) and (3.7), and separate the real and imaginary parts of functions P_n. We obtain

$$\rho\, c^2 P_n = a_n + i b_n \tag{3.8}$$

$$a_1 = 35\,u_1 + 10\,(u_2 + u_3) + u_4 + u_5,\ b_1 = 35\,u_6 + 10\,(u_7 + u_8) + u_9 + u_{10} \tag{3.9}$$

$$a_2 = -5\,u_1 - u_2 - u_3,\quad a_3 = u_1,\quad b_2 = -5\,u_6 - u_7 - u_8$$

$$b_3 = u_6$$

$$a_4 = -3\,u_2 - u_4,\ a_5 = u_2,\ b_4 = -3\,u_7 - u_9,\ b_5 = u_7$$

$$u_1 = w\left[\frac{1}{15} - \frac{1}{16}x^2\,(v + 2x^2)\right] + \frac{1}{16}(t_1 t_2 + t_3 t_4) \tag{3.10}$$

$$u_2 = \eta^2\left(\frac{1}{3} + \frac{1}{2}x^2\right) + \frac{1}{4}t_1 t_5,\quad u_3 = -\frac{1}{3} - \frac{1}{2}x^2 + \frac{1}{4}t_3 t_6$$

$$u_4 = \eta^4 u\,(w + x^2),\quad u_5 = -1 / (4 + x^2)$$

$$u_6 = \frac{1}{16}x^3\,(1 - \eta)\,(x^2 + 4 + 2\eta + 2\eta^2) + \frac{1}{48}x\,(4 - 3\eta + 2\eta^3 - 3\eta^5) + \frac{1}{64}(t_2 t_7 + t_4 t_8)$$

$$u_7 = -\frac{1}{2}x\eta^3 + \frac{1}{8}t_5 t_7,\quad u_8 = \frac{1}{2}x + \frac{1}{8}t_6 t_8,\quad u_9 = -2\eta^5 x u$$

$$u_{10} = 2 / [x\,(4 + x^2)],\ t_1 = -\operatorname{arctg}\,[2\,x / (x^2 - w)]$$

$$t_2 = x^7 + 3\,v x^5 + (1 + 2\,v + 3\,\eta^4)\,x^3 + w^2 v x,\ t_3 = \operatorname{arctg}\,(2 / x)$$

$$t_4 = x^7 + 6\,x^5 + 8\,x^3,\ t_5 = \eta^2\,x\,(v + x^2),\ t_6 = 2\,x + x^3$$

$$t_7 = \ln\,[1 + 4\,\eta / (x^2 + v - 2\,\eta)],\ t_8 = -\ln\,(1 + 4 / x^2)$$

$$v = 1 + \eta^2,\ w = 1 - \eta^2,\ 1 / u = x^4 + 2\,v x^2 + w^2$$

The integral $J^{(t)}_{nwrs}$ for transverse waves can be similarly calculated. However, a more direct way is to use formulas (2.9), since this enables us to utilize formulas (3.7) for longitudinal waves. The integral $J^{(t)}_{nwrs}$ is determined by formulas (3.1), (3.8), and (3.10), but now $x = x_t \equiv \omega a / c_t$ and $\eta = \eta_t \equiv c_t / c_l$, and it is necessary to substitute in formulas (3.9) $-u_1, -u_3, -u_2, -u_5, -u_4, -u_6, -u_8, -u_7, -u_{10}$, and $-u_9$ for $u_1, u_2, \ldots, u_{10}$.

The respective limit values for asymptotics of short and long waves can be determined by using the obtained general formulas and passing to limit in formulas (3.1), and (3.8)—(3.10). For asymptotics of short waves the series expansion in the small parameter $x = 1 / (qa) \ll 1$ yields for the integral J_{nwrs} the expression

$$\rho J^{(l)}_{nwrs} = \left[\frac{1}{c_t^2 - c_l^2} - \frac{5}{4c_l^2} + i\,\frac{a\omega}{2c_l^3}\right] k_{nwrs} + \left[\frac{1}{c_l^2} - \frac{1}{c_t^2}\right]\frac{\delta_{nwrs}}{15} + \left[\frac{1}{c_l^2 - c_t^2}\right]\theta_{nwrs} + \frac{\delta_{n(r}\delta_{s)w}}{3c_l^2} \tag{3.11}$$

$$\rho J^{(l)}_{nwrs} = \left[\frac{5}{4c_t^2} - \frac{1}{c_t^2 - c_l^2} - i\,\frac{a\omega}{2c_t^3}\right] k_{nwrs} + \left[\frac{1}{c_t^2} - \frac{1}{c_l^2}\right]\frac{\delta_{nwrs}}{15} + \left[-\frac{5}{4c_t^2} + i\,\frac{a\omega}{2c_t^3}\right]\theta_{nwrs} + \frac{\delta_{n(r}\delta_{s)w}}{3c_t^2}$$

For asymptotics of long waves the series expansion in the small parameter $1 / x = qa \ll 1$ yield the expression

$$J_{nwrs} = a^2\,\omega^2\,Y_{nwrs} + i a^3\,\omega^3\,Z_{nwrs} \tag{3.12}$$

$$\rho\,Y_{nwrs} = (3\,\varphi_{nwrs} - 2\,\delta_{nwrs})\,c^{-2}\,(c_t^{-2} - c_l^{-2}) / 105 - \theta_{nwrs} c^{-2} c_t^{-2} / 5 + \delta_{nwrs}\,(c_t^{-4} - c_l^{-4}) / 15 + \delta_{n(r}\delta_{s)w} c_t^{-2}\,(c^{-2} - 5\,c_t^{-2}) / 15$$

$$\rho Z_{nwrs} = \frac{2}{15}\,\delta_{nwrs}\,(c_l^{-5} - c_t^{-5}) + \frac{2}{3}\,\delta_{n(r}\delta_{s)w} c_t^{-5}$$

Integrals $J^{(l)}_{nwrs}$ and $J^{(t)}_{nwrs}$ are obtained from (3.12) by applying the respective substitutions $c \to c_l$ or $c \to c_t$.

All of the above reasoning is valid for quasi-isotropic polycrystals of arbitrary symmetry. Below, we present the calculation of scattering coefficients and ultrasonic wave propagation velocity in polycrystals of cubic symmetry.

4. The method of determining constants K_c and μ_c using a condition equivalent to (1.12) appeared in /11/. If the cubic symmetry tensors C and l are represented by their matrix coefficients $C_{11}, C_{12}, C_{44}, l_{11}, l_{12}$, and l_{44}, then in conformity with (1.6), the anisotropy parameter h

$$h = l_{11} - l_{12} - 2\,l_{44} = C_2\,[(1 + \varkappa\,C_1)\,(1 + \varkappa\,C_1 + \varkappa\,C_2)]^{-1} \tag{4.1}$$
$$C_1 \equiv 2\,C_{44} - 2\,\mu_c,\quad C_2 \equiv C_{11} - C_{12} - 2\,C_{44},\quad \varkappa = 3\,(K_c + 2\,\mu_c)\,[5\,\mu_c\,(3\,K_c + 4\,\mu_c)]^{-1}$$

We introduce the symbolic notation form for the tensor function of the fourth rank in the isotropic space /13/

$$(E_1, E_2, E_3, E_4, E_5)_{nwrs} \equiv E_1 k_{nwrs} + E_2\,(\delta_{nr}\,k_{ws} + \delta_{ns}\,k_{wr} + \delta_{ws} k_{nr} + \delta_{wr} k_{ns}) - E_3\,(\delta_{nw} k_{rs} + \delta_{rs} k_{nw}) + E_4\,(\delta_{nr}\delta_{ws} + \delta_{ns}\delta_{wr}) - E_5\,\delta_{nw}\delta_{rs} \tag{4.2}$$

We substitute the obtained expression for the integral J_{nwrs} and for the convolution of the covariant tensor of the polycrystal of cubic symmetry /14/

$$A^{nwuu}_{rsvv} = A^{nw11}_{rsuu} = 0,\quad A^{nwuv}_{rsuv} = 21\,\xi\,(0, 0, 0, 3, 2)_{nwrs} \tag{4.3}$$
$$A^{nw1u}_{rs1u} = 3\,\xi\,(0, 3, 4, 5, 2)_{nwrs},\quad A^{nw11}_{rs11} = \xi\,(1, 5, 7, 5, 1)_{nwrs},\qquad \xi \equiv h^2 / 525$$

into formula (2.1). We have

$$l_{*nwrs} = \xi\,(P_1,\ 5\,P_1 + 9\,D,\ 7\,P_1 + 12\,D,\ 5\,P_1 + 15\,D + 63\,D',\ P_1 + 6\,D + 42\,D')_{nwrs},\quad D \equiv 4\,P_2 + P_4,\qquad D' \equiv 2\,P_3 + P_5 \tag{4.4}$$

Examining the conditions of existence of mean fields in the form of plane waves, we come to the characteristic equation /11/

$$\det\,[\rho\omega^2\delta_{nr} - q_*^2 k_{*ws}\overline{C}_{*nwrs}] = 0 \tag{4.5}$$

where $k_{*n} = q_{*n} / q_*$ and $\mathbf{q}_*$ is a complex "wave vector". The three roots of Eq. (4.5), which are cubic relative to the square of the wave vector, make it possible to determine the correlation correction $\overline{C}'_{*nwrs}$ contribtution to scattering. Let

$$\overline{C}'_{*nr} = k_{*ws}\overline{C}'_{*nwrs} = (\mathrm{N}^{(l)} - \mathrm{N}^{(t)} + i\mathrm{M}^{(l)} - i\mathrm{M}^{(t)})\,k_{*wr} + (\mathrm{N}^{(t)} + i\mathrm{M}^{(t)})\,\delta_{wr} \tag{4.6}$$

Then, assuming the correlation correction to be small, we obtain

$$\gamma_{l,t}(\omega) = \frac{\omega \mathrm{M}^{(l,t)}(\omega)}{2\rho c^3_{l,t}} \tag{4.7}$$

$$v_{l,t}(\omega) = c_{l,t}\left[1 + \frac{1}{2\rho c^2_{l,t}}\left(1 + \omega\frac{d}{d\omega}\right)\mathrm{N}^{(l,t)}(\omega)\right] \tag{4.8}$$

where γ is the scattering coefficient normalized with respect to a unit of length, and formula (4.8) for wave dispersion velocity $v(\omega)$ is defined by the relation

$$v = (d\,\chi / d\,\omega)^{-1},\quad q_* = \chi - i\,\gamma$$

5. After necessary calculations using formulas (1.14) and (4.4)—(4.7), we obtain for the scattering coefficient

$$\gamma_l = \frac{\omega}{3\rho c_l^3}\,\frac{m''}{(\varkappa m'')^2 + |1 + \varkappa m'|^2},\quad \gamma_t = \frac{\omega}{4\rho c_t^3}\,\frac{n''}{(\varkappa n'')^2 + |1 + \varkappa n'|^2} \tag{5.1}$$
$$m' = \xi'\,(2\,u_1 - 2\,u_2 + 4\,u_3 - 5\,u_4 + 4\,u_5),\quad n' = 2/3\,m' + \xi'\,(2\,u_5 - u_3)$$
$$m'' = \xi'\,(2\,u_6 - 2\,u_7 + 4\,u_8 - 5\,u_9 + 4\,u_{10}),\quad n'' = 2/3\,m'' + \xi'\,(2\,u_{10} - u_8)$$
$$\xi' = 2\,h^2 / (175\,\rho\,c^2)$$

where u_n, h, and $\varkappa$ are defined by Eqs. (3.10) and (4.1).

To determine the scattering coefficients γ_l and γ_t of longitudinal and transverse waves, respectively, it is, thus, necessary to use the first and second of Eqs. (5.1). For passing from parameter x to the wave number q formula (2.4) is used. All quantities pertaining to a particular wave form must bear the subscript l or t. We stress the necessity to use the subscripts, since $c_l \neq c_t$, $x_l \neq x_t$, $\eta_l = 1 / \eta_t = c_l / c_t$, etc.

For scattering coefficients (5.1) we obtain three asymptotics. When $m', m'', n', n'' \ll 1/\varkappa$, it is possible to equate the denominators in (5.1) to unity, and two cases are possible, viz. $x \gg 1$ or $x \ll 1$. In the first case, using asymptotics (3.12) we obtain for long waves

$$\gamma_l = \frac{4h^2a^3\omega^4}{375\rho^2c_l^3}\left(\frac{2}{c_l^5}+\frac{3}{c_t^5}\right), \quad \gamma_t = \frac{h^2a^3\omega^4}{125\rho^2c_t^3}\left(\frac{2}{c_t^5}+\frac{3}{c_l^5}\right) \tag{5.2}$$

In the second case, using formulas (3.11), we have

$$\gamma_l = \frac{4h^2a\omega^2}{525\rho^2c_l^6}, \quad \gamma_t = \frac{h^2a\omega^2}{150\rho^2c_t^6} \tag{5.3}$$

When $m', m'', n', n'' \gg 1/\varkappa$ we have $x \ll 1$ and, as implied by (5.1), (3.10), and (3.11), the expressions in brackets in formulas (5.1) can be neglected. For short waves this yields

$$\gamma_l = \frac{175}{12h^2\varkappa^2a}, \quad \gamma_t = \frac{75}{8h^2\varkappa^2a} \tag{5.4}$$

Formulas (5.2) and (5.3) differ from those for asymptotics of single scattering in /2/ by that in them the calculation of the anisotropy parameter h takes into account the dependence of auxilliary elastic constants on the choice of the reference body. In asymptotics (5.4) parameters h and $\varkappa$ are affected by the selection of the reference body.

6. Using formulas (1.14), (3.8)—(3.10), (4.4)—(4.6), and (4.8) for calculating the propagation velocities v_l and v_t of longitudinal and transverse waves, we obtain

$$v_l = c_l\left[1 - \frac{1}{3\varkappa\rho c_l^2}\left(\frac{\eta_1 - x\eta_3}{\eta_1^2+\eta_2^2} + \frac{2x(\eta_1^2\eta_3+\eta_1\eta_2\eta_4)}{(\eta_1^2+\eta_2^2)^2} - 1\right)\right] \tag{6.1}$$

$$v_t = c_t\left[1 - \frac{1}{4\varkappa\rho c_t^2}\left(\frac{\eta_5 - x\eta_7}{\eta_5^2+\eta_6^2} + \frac{2x(\eta_5^2\eta_7+\eta_5\eta_6\eta_8)}{(\eta_5^2+\eta_6^2)^2} - 1\right)\right] \tag{6.2}$$

where

$$\begin{aligned}
&\eta_1 = \tau(2u_1 - 2u_2 + 4u_3 - 5u_4 + 4u_5) + 1 \qquad (6.3)\\
&\eta_2 = \tau(2u_6 - 2u_7 + 4u_8 - 5u_9 + 4u_{10})\\
&\eta_3 = \tau(2H_1 - 2H_2 + 4H_3 - 5H_4 + 4H_5)\\
&\eta_4 = \tau(2H_6 - 2H_7 + 4H_8 - 5H_9 + 4H_{10})\\
&\eta_5 = 2/3\,\eta_1 + 1/3 + \tau(2u_5 - u_3), \quad \eta_6 = 2/3\,\eta_2 + \tau(2u_{10} - u_8)\\
&\eta_7 = 2/3\,\eta_3 + \tau(2H_5 - H_3), \quad \eta_8 = 2/3\,\eta_4 + \tau(2H_{10} - H_8)\\
&H_1 = -\frac{1}{8}xw(v+4x^2) + \frac{1}{32}(t_1w_2 + w_1t_2 + t_3w_4 + w_3t_4)\\
&H_2 = x\eta^2 + \frac{1}{4}(t_1w_5 + w_1t_5), \quad H_3 = -x + \frac{1}{4}(t_3w_6 + w_3t_6)\\
&H_4 = -2x\eta^4(x^4 + 2wx^2 + 2vw - w^2)u^2, \quad H_5 = 2x(4+x^2)^{-2}\\
&H_6 = \frac{1}{16}x^2(1-\eta)(5x^2 + 12 + 6\eta + 6\eta^2)\\
&\qquad \frac{1}{48}(4 - 3\eta - 2\eta^3 - 3\eta^5) + \frac{1}{64}(t_7w_2 + w_7t_2 + t_8w_4 + w_8t_4)\\
&H_7 = -\frac{1}{2}\eta^3 + \frac{1}{8}(t_7w_5 + w_7t_5)\\
&H_8 = 1/2 + 1/8\,(t_8w_6 + w_8t_6), \quad H_9 = 2\eta^5(3x^4 + 2vx^2 - w^2)u^2\\
&H_{10} = -2(3 + 4x^{-2})(x^2+4)^{-2}, \quad w_1 = (x^2 - \eta^2 + 1)u\\
&w_2 = 7x^6 + 15vx^4 + 3(1 + 2v + 3\eta^4)x^2 + w^2v, \quad w_3 = 2u_5\\
&w_4 = 7x^6 + 30x^4 + 24x^2, \quad w_5 = 3x^2\eta^2 + v\eta^2, \quad w_6 = 2 + 3x^2\\
&w_7 = 8x\eta u, \quad w_8 = 4u_{10}, \quad \tau = 2\varkappa h^2/(175\rho c^2)
\end{aligned}$$

The quanties w, v, u, and u_n are defined by formulas (3.10), and h and $\varkappa$ by formulas (4.1).

Thus for the calculation of propagation velocities v_l and v_t of longitudinal and transverse waves formulas (6.1) and (6.2) are to be used, respectively. All quantities must bear subscript l or t. Formula (2.4) is used for passing from parameter x to the wave number q.

Formulas for wave propagation velocities can be obtained in longwave approximation using formulas (2.54) and (2.55) in /11/ by carrying out the substitution $\gamma_6 - h$ and setting $a_1 = a_3 = 0$. The short wave dispersion velocity is slight.

The dependence of dimensionless scattering coefficients on frequency and size of the inhomogeneity grain calculated by the derived here formulas for the general case and, also, for that of asymptotics of long and short waves are shown in Fig.1. The calculation relate to copper whose reference body elastic constants and moduli, expressed in units of 10^{10} N/m^2, were taken to be: $C_{11} = 16.905$, $C_{12} = 12.193$, $C_{44} = 7.550$, $K_c = 13.76$, $\mu_c = 4.87$ /15/, with the density $\rho = 8960\ \mathrm{kg/cm}^3$. The correlation scale a was taken equal to the grain mean diameter $\bar{D}$ /1/.

The numerals 1 and 2 denote (in Fig.1) the scattering coefficients $\gamma_l(x_l)$ and $\gamma_t(x_t)$, respectively, for longitudinal and transverse waves. The dash lines relate to γ_l and γ_t calculated by asymptotic formulas for long and short waves. The quantities $1/x_l$ and $1/x_t$ for longitudinal and transverse waves are plotted on the axis of abscissas.

In the high- and low-frequency regions (large and small inhomogeneity grains) $\gamma_l > \gamma_t$, and in the intermediate region $\gamma_l < \gamma_t$. It will be seen that when the wave length λ considerably exceeds the grain dimension D, the coefficients of ultrasonic wave scattering by polycrystal grains are proportional to $\bar{D}^3\omega^4$ (the Rayleigh scattering region), at lower values of $\lambda/\bar{D}$ they are proportional to $\bar{D}\omega^2$ (the phase scatter in region), and when $\lambda/\bar{D} \ll 1$ they are inversely proportional to the grain size (the diffusion scattering region). A similar dependence is obtained also when the calculation is based on energy flux density at the grain boundary /16/. This dependence is, moreover, confirmed experimentally.

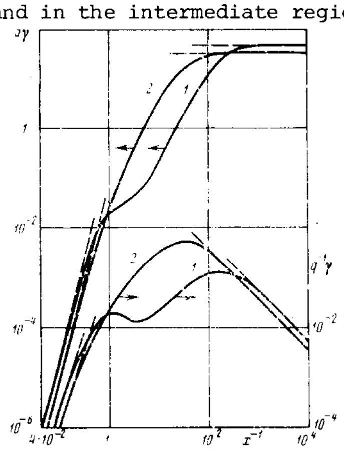

Fig.1

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